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DYNAMIC PRICING WITH CAPACITY CONSTRAINTS AND INVENTORY REPLENISHMENT

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Abstract. This paper describes a fast algorithm for solving a capacitated dynamic pricing problem where the producer has the ability to store inventory. The pricing problem described is a quadratic programming problem with a structure that can be solved effectively by a dual algorithm. The proposed algorithm gives a solution satisfying the Karush-Kuhn-Tucker conditions. This, combined with the fact that the problem has a convex feasible region with a concave objective function which we want to maximize, implies that the proposed algorithm gives a globally optimal solution. The algorithm is illustrated by numerical examples for both the single-item and the multi-item cases.

1. Introduction

1.1. Problem

While the rich literature on production management typically assumes that product prices are exogenous to the production management problem, the jointly dynamic pricing and production problem concerns the decisions about which prices to charge the customers over time in order to maximize company revenue. Although such pricing models have been studied since the 1960's, their adoption by industrial practice seems to be limited to *revenue management* (see e.g. [10]), within pure service systems where inventory replenishments are impossible due to perishable resources. The benefits of revenue management within service industries such as airlines, hotels and electric utilities are well-known, and should today indicate an increased adoption of models that coordinate demand management and operation management in industries characterized by inventory replenishments.

This paper is concerned with the problem of integrated pricing and production decisions involving a single monopoly firm producing non-perishable products that can be replenished throughout the planning horizon. The demand is assumed to be deterministic, price-dependent and non-stationary. The planning takes place in a discrete-time framework, where the operational costs of inventory and production smoothing, due to capacity constraints, are assumed to play an important role. It is assumed that no set-up costs occur, and the demand functions are linear.

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1.2. Literature

The research progress within revenue management has been comprehensive. Reviews of this literature are found in McGill and van Ryzin [9] and Boyd and Bilegan [2]. A monograph on the topic was written by Phillips [10]. In recent years, a growth has also been observed in the literature concerning integrated pricing and production decisions where products are replenished. An overview of this literature can be found in Eliashberg and Steinberg [4] and, more recently, in review articles by Elmaghraby and Keskinoack [5], Chan et al. [3] and Yano and Gilbert [15].

Wagner and Whitin (1958a) [13] introduce price as a decision variable in a discrete-time-model for a monopolist that produces a single product with unlimited capacity. Under the assumption of constant marginal costs, they show that their well-known planning horizon results (1958b) [14] are still valid in cases with price-dependent demands. Thomas [12] extends the Wagner and Whitin (1958a) model for the uncapacitated case to a situation where the set-up costs, variable production costs and inventory holding costs may differ from period to period. Thomas shows the existence of planning horizons for this problem and formulates a fast and efficient dynamic programming algorithm for the problem. Bhattacharjee and Ramesh [1] develop a fast heuristic search procedure to solve the uncapacitated model where the demand is assumed to have a constant elasticity of demand.

Kunreuter and Schrage [8] and Gilbert [7] have studied a variant of the uncapacitated single-product pricing problem where a *fixed* price must be chosen for the entire planning period. Gilbert [6] extends his model to jointly determining fixed prices and production schedules in the multi-item case with capacity constraints but without set-up costs. Swann [11] analyzes a single-item pricing and production problem with capacity constraints and no set-up costs and shows that a greedy algorithm provides an optimal solution for the pricing and production problem. Swan reports that, in some cases, the proposed greedy algorithm gives an optimal solution to the multiple product problem.

1.3. Motivation

We propose a fast algorithm for solving the joint inventory and pricing problem in a production environment with multiple products and capacity constraints. Like in Gilbert [7] and Swan [11], the proposed algorithm assumes zero set-up costs (and set-up times). The motivation for studying production systems with zero set-up costs and times is based on the last decades' expansions in so-called "Flexible Manufacturing Systems" (FMSs). Modern FMSs typically consist of robots with high ability to shift capacity quickly from one product to another at a low cost. Increased customization and broader product variety will most certainly lead to further improvements in production systems without major set-up (or retooling) costs. We also believe that increased customization and improvements in FMS technologies will call for production planning methodologies that can handle pricing decisions (dynamic pricing) for hundreds (or even thousands) of products within a dynamic environment, characterized with a short planning horizons (few planning periods) and a short "life-cycle" for products. The intention of the proposed algorithm is to fit into such a production environment.

2. The pricing problem

We consider a combined pricing and production problem over a finite horizon T. For each period $t = 1, \ldots, T$, the producer must decide the price per unit of product $j = 1, \ldots, J$, the production quantities, and the number of units of inventory kept from one period to another in order to maximize the total profit over the time horizon. The price functions are assumed to be linear, i.e.,

$$p_{jt} = \alpha_{jt} - \beta_{jt} d_{jt}, \qquad \forall j, t \leq T. \tag{2.1}$$

The demands might be time-varying and the production is assumed to be capacity limited. The problem can be stated as

$$\max_{d,x,i} \Pi = \sum_{j=1}^{J} \sum_{t=1}^{T} [(\alpha_{jt} - \beta_{jt} d_{jt}) d_{jt} - h_{jt} i_{jt} - v_{jt} x_{jt}] \tag{2.2}$$

s.t.

$$x_{jt} + i_{j,t-1} - i_{jt} = d_{jt}, \qquad \forall j, t \leq T, \tag{2.3}$$

$$\sum_{j=1}^{J} x_{jt} \leq r_t, \qquad \forall t \leq T, \tag{2.4}$$

$$x_{jt}, d_{jt}, i_{jt} \geq 0, \qquad \forall j, t \leq T, \tag{2.5}$$

where the *decision variables* are

x_{jt} = amount of product j to be produced in period t,
d_{jt} = demand of product j in period t,
i_{jt} = inventory of product j carried from period t to period $t+1$,

and *problem data* are

T = time horizon,
i_{j0} = starting inventory of product j carried from period 0 to period 1,
r_t = maximal production capacity in period t,
h_{jt} = unit inventory holding cost of product j in period t,
v_{jt} = unit production cost product j in period t,
α_{jt}, β_{jt} = parameters in the linear demand function in period t, $\alpha_{jt} > 0$, $\beta_{jt} > 0$.

We will refer to the problem (2.2)–(2.5) as the *original problem* P.

The objective (2.2) is to maximize the difference between sales revenue and the production and inventory costs over the time horizon T. The inventory balance equations (2.3) imply that no backlogging is allowed. Constraints (2.4) state that the facility has the capacity to produce up to r_t units in period t, i.e., for simplicity (and without loss of generality), we assume that producing one product unit of each product j consumes one capacity unit. Constraints (2.5) are non-negativity constraints on the decision variables. Note that the non-negativity constraints on demands imply that $p_{jt} \leq \alpha_{jt}$, $\forall j, t$.

All the constraints are linear and, therefore, problem (P) defines a convex feasibility set. The objective function is concave and, hence, the problem is a concave maximization problem.

Problem (P) can be solved by standard algorithms for quadratic programming. However, as it will be shown, the problem has a special structure, which can be utilized in a faster specialized dual algorithm.

2.1. The disaggregated problem

Problem (P) can be reformulated by introducing disaggregated production variables. This problem is denoted as (DP) and is stated as follows:

$$\max_x \Pi = \sum_{j=1}^{J}\sum_{t=1}^{T}\left[\left(\alpha_{jt} - \beta_{jt}\sum_{i=1}^{t} x_{jit}\right)\sum_{i=1}^{t} x_{jit} - \sum_{i=1}^{t}\left(h_{jit} + v_{ji}\right) x_{jit}\right] \tag{2.6}$$

s.t.

$$\begin{aligned} \sum_{j=1}^{J}\sum_{t=i}^{T} x_{jit} \quad &\leq \quad r_i, \qquad \forall i \leq T, &(2.7)\\ x_{jit} \quad &\geq \quad 0, \qquad \forall j, i, t \leq T, \quad i \leq t, &(2.8)\end{aligned}$$

where the *decision variables* are

x_{jit} = part of demand of product j in period t produced in period $i, i \leq t$.

We have the additional *problem data*

h_{jit} = unit inventory holding cost from period i to period t, $i \leq t$, product j.

The objective (2.6) maximizes the difference between sales revenue and the production and inventory costs over the time horizon. Constraints (2.7) are the production capacity constraints. Constraints (2.8) are the positivity constraints on the disaggregated production variables. Note that the non-negativity assumption for the disaggregated production variables ensures that the demands and inventories are non-negative. Non-negative demands imply $p_{jt} \leq \alpha_{jt}$, $\forall j, t$.

We have the following identities between the decision variables in (P) and (DP) that verify that problem (P) can be transformed into problem (DP) and vice versa:

$$\begin{aligned} x_{ji} &= \sum_{t=i}^{T} x_{jit}, && \forall j, i,\\ p_{jt} &= \alpha_{jt} - \beta_{jt}\sum_{i=1}^{t} x_{jit}, && \forall j, t,\\ d_{jt} &= \sum_{i=1}^{t} x_{jit}, && \forall j, t,\end{aligned}$$

$$i_{jt} = \sum_{\tau=t+1}^{T} \sum_{i=1}^{\tau} x_{ji\tau}, \qquad \forall j, t.$$

In addition, we assume that the inventory costs are additive over the periods, i.e.,

$$h_{jit} = \sum_{k=i}^{t-1} h_{jk}, \qquad \forall j, t.$$

The problem DP has $JT(T+1)/2$ disaggregated variables.

2.2. The Karush-Kuhn-Tucker conditions

The problem DP has a quasi-concave objective function and all the constraint functions are linear and, therefore, convex. Thereby the problem is reduced to finding a *solution* that satisfies the *Karush-Kuhn-Tucker conditions.*

By forming the partial *Lagrangian function* of the disaggregated problem (2.6)–(2.8):

$$\begin{aligned} L \;=\; & \sum_{j=1}^{J}\sum_{t=1}^{T}\left[\left(\alpha_{jt} - \beta_{jt}\sum_{i=1}^{t} x_{jit}\right)\sum_{i=1}^{t} x_{jit} - \sum_{i=1}^{t}\left(h_{jit}+v_{ji}\right)x_{jit}\right] - \\ & -\sum_{i=1}^{T}\lambda_i\left(\sum_{j=1}^{J}\sum_{t=i}^{T} x_{jit} - r_i\right), \end{aligned}$$

the Karush-Kuhn-Tucker conditions for this problem can be stated as

$$\begin{aligned} \alpha_{jt} - 2\beta_{jt}d_{jt} - (v_{ji} + h_{jit} + \lambda_i) &\le 0, \quad \forall j, i, t, i \le t, && (2.9)\\ \left[\alpha_{jt} - 2\beta_{jt}d_{jt} - (v_{ji} + h_{jit} + \lambda_i)\right] x_{jit} &= 0, \quad \forall j, i, t, i \le t, && (2.10)\\ \sum_{j=1}^{J}\sum_{t=i}^{T} x_{jit} - r_i &\le 0, \quad \forall i, i \le t, && (2.11)\\ \left(\sum_{j=1}^{J}\sum_{t=i}^{T} x_{jit} - r_i\right)\lambda_i &= 0, \quad \forall i, i \le t, && (2.12)\\ x_{jit} &\ge 0, \quad \forall j, i, t, i \le t, && (2.13)\\ \lambda_i &\ge 0, \quad \forall i. && (2.14) \end{aligned}$$

The multipliers λ_i express the marginal increase in profit of one extra capacity unit in period i. The expression $v_{ji} + h_{jit} + \lambda_i$ is the *opportunity cost* of producing one unit of product j in period i and keeping this unit stored into period t (or if $i = t$, the opportunity cost of producing product j in period t). The expression $\alpha_{jt} - 2\beta_{jt}d_{jt}$ is the *marginal revenue* from the demand of product j in period t. Conditions (2.9) and (2.10) state that, if production of product j takes place in period i in order to cover demand in period t, that is $x_{jit} > 0$, then the opportunity cost of this production must be equal to the marginal revenue in period t. On the other hand, if the opportunity cost of producing one unit in i and storing this unit into t exceeds the marginal revenue in t, then x_{jit} must be zero. Conditions (2.11) are the capacity constraints of the problem while equations (2.12) state that if the Lagrangian multiplier in a period i is strictly positive, $\lambda_i > 0$, then the capacity in period i is fully utilized. The conditions (2.13) and (2.14) are non-negativity constraints on the multiplier values and the decision variables.

To simplify the notation, let γ_{jit} denoted the opportunity cost of producing the demand for product j in period t in period i, i.e.,

$$\gamma_{jit} = v_{ji} + h_{jit} + \lambda_i, \qquad \forall j, i, t, i \leq t.$$

By this, the KKT-condition (2.9) can be written as

$$\alpha_{jt} - 2\beta_{jt} d_{jt} - \gamma_{jit} \leq 0, \qquad \forall j, i, t, i \leq t, \tag{2.15}$$

and (2.10) as

$$[\alpha_{jt} - 2\beta_{jt} d_{jt} - \gamma_{jit}]\, x_{jit} = 0, \qquad \forall j, i, t, i \leq t. \tag{2.16}$$

Since the expression $\alpha_{jt} - 2\beta_{jt} d_{jt}$ (the marginal revenue product j in period t) is independent of the production period, i.e., i or i', $i, i' \leq t$, $i \neq i'$ that cover the demand in period t, it follows that (2.15) and (2.16) alternatively can be stated by (2.17)–(2.20) in the following proposition. (The proposition is only a reformulation of the above KKT-conditions and is, therefore, given without any proof).

Proposition 2.1. (KKT-conditions) *If x is an optimal solution of problem DP with non-negative numbers λ_i, $\left(\sum_{j=1}^{J} \sum_{t=i}^{T} x_{jit} - r_i \right) \lambda_i = 0$, $\forall i \leq T$, then, for $\forall j, i, i', i' \leq t$, $i \neq i'$,*

$$x_{ji't} > 0,\ x_{jit} > 0 \ \Rightarrow\ \gamma_{jit} = \gamma_{ji't} = \alpha_{jt} - 2\beta_{jt} d_{jt}, \tag{2.17}$$

$$x_{ji't} > 0,\ x_{jit} = 0 \ \Rightarrow\ \gamma_{jit} \geq \gamma_{ji't} = \alpha_{jt} - 2\beta_{jt} d_{jt}, \tag{2.18}$$

$$x_{ji't} = 0,\ x_{jit} > 0 \ \Rightarrow\ \gamma_{ji't} \geq \gamma_{jit} = \alpha_{jt} - 2\beta_{jt} d_{jt}, \tag{2.19}$$

$$x_{ji't} = 0,\ x_{jit} = 0 \ \Rightarrow\ \big(\alpha_{jt} - 2\beta_{jt} d_{jt} \leq \gamma_{jit} \text{ and } \alpha_{jt} - 2\beta_{jt} d_{jt} \leq \gamma_{ji't}\big). \tag{2.20}$$

Proposition 2.1, and especially condition (2.17), is important for the algorithmic design of the proposed algorithm. Conditions (2.18) and (2.19) state that the demand for product j, period t in an optimal plan will be produced in the period with the *lowest opportunity cost* to cover that demand. Condition (2.17) says that, if an optimal plan suggests that the demand for a certain period is covered from more than one production period, then the opportunity costs of covering this demand from these periods must be *equal.* By (2.20), it follows that if the opportunity costs of producing the demand in period t in periods i and i' exceed the marginal revenue, then no production takes place in these two periods in order to cover the demand for product j in period t.

2.3. The algorithmic idea

The proposed algorithm to solve DP is designed to find a point (x^*, λ^*) that satisfies the set of KKT-conditions (2.9)–(2.14) where the conditions (2.9) and (2.10) by Proposition 2.1 can be substituted with (2.17) into (2.20). To find such a point is complicated because of the capacity constraints, but if we relax them by setting $\lambda^0 = 0$, then it is easy to find a point $(x^0, \lambda^0 = 0)$, i.e., an initial solution where the demand for each period is covered by the production in the periods that have the lowest costs (variable production costs and inventory costs). Unless this solution is feasible, at which the algorithm terminates, a *multiplier adjustment procedure* is conducted in order to remove infeasibilities iteratively. Each iteration k will define a *transformation* $\Gamma^k : (x^{k+1} = x^k + \Delta x^k,\ \lambda^{k+1} = \lambda^k + \Delta\lambda^k)$, $\Delta\lambda^k \geq 0$, with the property that it will *decrease* the capacity overload in periods with capacity overload *without* causing capacity overload in the other periods, and such that the conditions (2.17) to (2.20) in the proposition are satisfied.

To be more accurate, let the *capacity overload* $\overline{r_i}$ in period i be defined as

$$\overline{r_i} = \max\left\{\sum_{j=1}^{J}\sum_{t=i}^{T} x_{jit} - r_i, 0\right\}, \qquad \forall i \leq T,$$

while, on the other hand, the *excess capacity* $\underline{r_i}$ in period i is defined as above

$$\underline{r_i} = \max\left\{r_i - \sum_{j=1}^{J}\sum_{i=t}^{T} x_{jit}, 0\right\}, \qquad \forall i \leq T.$$

Then, for a given (x, λ), the set of all periods with capacity overload is given by

$$\overline{R}_i = \{i \leq T : \overline{r_i} > 0\},$$

while the remaining periods with possibly excess capacity is given by

$$\underline{R}_i = \left\{i \leq T : \underline{r_i} \geq 0\right\}.$$

With these definitions, a *skeleton* of the algorithm can now be formulated.

ALGORITHM (SKELETON)

Start (x^0, λ^0), $\lambda^0 = 0$.

While $\overline{R}_i^k \neq \emptyset$ **do**

Define Γ^k with

1) $\lambda^{k+1} = \lambda^k + \Delta\lambda^k$, $\Delta\lambda^k \geq 0$,

2) $x^{k+1} = x^k + \Delta x^k$, $x^{k+1} \geq 0$,

such that

3) $0 \leq \overline{r_i}^{k+1} < \overline{r_i}^k \quad \forall i \in \overline{R}_i^k$,

4) $0 \leq \underline{r_i}^{k+1} \leq \underline{r_i}^k \quad \forall i \in \underline{R}_i^k$,

5) $\Delta\lambda_i^k > 0$ if $\overline{r_i}^k \geq 0$,

6) x^{k+1} satisfies (2.17)–(2.20) in Proposition 2.1.

The skeleton says that, unless there are no more periods with capacity overload, at each iteration k, we get closer to a feasible solution by applying a transformation Γ^k defined by 1) and 2) such that 3) the capacity overload is decreased in periods i with capacity overload, and 4) the capacity overload in all periods with possibly excess capacity remains zero. 5) says that the multiplier for a period i is increased only if the capacity in this period is balanced or overloaded. Note that the multiplier in a period with balanced capacity is allowed to be increased. If so happens, the increase in the multiplier is motivated by liberating capacity in order to produce into a period with capacity overload. 6) says that conditions (2.17) to (2.20) in Proposition 2.1 must be satisfied at each stage of the algorithm.

If we are able to construct such a transformation as described above, it should be evident that the algorithm terminates with a solution that satisfies the complete set of KKT-conditions (2.9)–(2.14). The difficult part of the algorithm is the construction of Γ^k that satisfes the KKT-conditions (2.9) and (2.10), which is the subject of the next section.

2.4. Construction of the algorithm

2.4.1. Initialization. As stated in the previous section, the proposed algorithm to solve the disaggregated problem starts with relaxing the capacity constraints of the problem. By setting all the multipliers equal to zero in the problem, it follows by Proposition 2.1 that each demand will be covered from the production period that covers this demand at the lowest opportunity cost. When all multipliers are set equal to zero, the opportunity costs consist of variable production costs and inventory holding costs.

To illustrate the initialization step, consider the following simple three-periodic, single-product example where the price-functions and costs are equal for the three periods, but where the capacity differs. The unit production cost is 20 and the inventory holding cost is 2.

t	1	2		3		
p_t	$p_1 = 100 - 1d_1$	$p_2 = 100 - 1d_2$		$p_3 = 100 - 1d_3$		
x_{it}	x_{11}	x_{12}	x_{22}	x_{13}	x_{23}	x_{33}
$\gamma_{it} = v_i + h_{it}$	20	22	20	24	22	20
r_i	50	10		21		

Table 1. A three-periodic, single product problem (problem data).

In Table 2.4.1, row 2 shows the linear price-functions and row 3 the possible disaggregated production variables that can cover the demand in each period. Row 4 shows the unit costs associated with each disaggregated production variable, while row 5 gives the capacity constraints in the problem. If capacity is unconstrained, Table 2.4.1 shows that it will be most profitable to produce the demands in the same period where demand occurs (at a cost of 20).

Initially, the capacity constraints are relaxed by setting the multipliers equal to zero and optimal prices and demands are calculated.

From the KKT-condition (2.9) and (2.10), it follows directly that, if $x_{tt} > 0$, then

$$\alpha_t - 2\beta_t d_t - (v_t + \lambda_t) = 0.$$

With $\lambda_t = 0$, the optimal demands d_t at the initialization step will be

$$\alpha_t - 2\beta_t d_t - v_t = 0,$$

$$\Rightarrow d_t^0 = \frac{\alpha_t}{2\beta_t} - \frac{v_t}{2\beta_t}. \tag{2.21}$$

Optimal prices are found by substituting (2.21) into the linear price-functions (2.1) which gives

$$p_t^0 = \frac{\alpha_t}{2} + \frac{v_t}{2}. \tag{2.22}$$

The optimal demand when $\lambda_t > 0$ is similarly calculated to be

$$d_t^0 = \frac{\alpha_t}{2\beta_t} - \frac{v_t}{2\beta_t} - \frac{\lambda_t}{2\beta_t}. \tag{2.23}$$

From (2.21) and (2.23), it can easily be seen that an increase in λ_t by one unit ($\Delta\lambda_t = 1$) reduces the demand d_t by $\frac{1}{2\beta_t}$. Since we have assumed that each unit produced consumes one capacity unit (see (2.4)), it follows that a *specified increase* in the multiplier of $\Delta\lambda_t$ leads to a reduction in capacity overload, $\Delta\overline{r}_t$, in period t of

$$\Delta\overline{r}_t = \frac{1}{2\beta_t}\Delta\lambda_t. \tag{2.24}$$

By (2.24), it follows that the necessary increase $\Delta\lambda_t$ in order to achieve a *specified* reduction in capacity overload in period t is

$$\Delta\lambda_t = \frac{\Delta\overline{r}_t}{\frac{1}{2\beta_t}} = \Delta\overline{r}_t \cdot 2\beta_t. \tag{2.25}$$

The expressions (2.24) and (2.25) are calculated for later use while (2.21) and (2.22) are used to calculate optimal prices and demands at the initialization step of our three-periodic example. The results of these calculations are shown in Table 2. The profit contribution from the plan is omitted in the table, but can easily be calculated by (2.2).

t	1	2	3
p_t	60	60	60
d_t	40	40	40
x_{it}	$x_{11} = 40$	$x_{22} = 40$	$x_{33} = 40$
r_i	50	10	21
$\overline{r}_i$,$\underline{r}_i$	$\underline{r}_i = 10$	$\overline{r}_2 = 30$	$\overline{r}_3 = 19$

Table 2. Initialization step (stage 0).

For later use, we introduce the graph in Figure 1 to illustrate each step of the algorithm.

In the figure, each circle represents a disaggregated variable where the rows correspond to demand periods while the columns correspond to the production periods, i.e., the circle in row 2, column 1 corresponds to the disaggregated variable x_{12}. Each circle is divided by a line where the number above the line is the value of the disaggregated variable, while the number below the line is the value of the corresponding opportunity cost. The top of the figure shows the multiplier values and the capacity overloads or excess capacities at each step of the algorithm.

As can be seen from Figure 1, the initial solution has capacity overload in period 2 and 3, but has excess capacity in period 1. Many options exist in order to remove

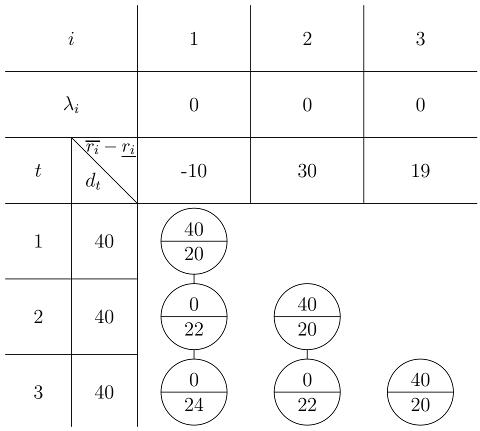

Figure 1. (Step 0): Initialization, $\lambda = 0$.

the capacity overload in period 2 and 3. One way is to the increase the multipliers (and thereby the opportunity costs) in these two periods up to a level that exactly removes the overloads, but as we will see, this solution will not satisfy the KKT-conditions. The necessary increases in the multipliers to remove capacity overloads of 30 and 19 period 2 and 3 respectively are calculated by equation (2.25), which gives the values

$$\Delta\lambda_2 = 30 \cdot 2 \cdot 1 = 60,$$
$$\Delta\lambda_3 = 19 \cdot 2 \cdot 1 = 38,$$

and opportunity costs

$$\gamma_{22} = 20 + 60 = 80, \tag{2.26}$$
$$\gamma_{33} = 20 + 38 = 58. \tag{2.27}$$

The demands for the two periods are calculated as

$$d_2 = \frac{\alpha_2}{2\beta_2} - \frac{\gamma_{22}}{2\beta_2} = \frac{100}{2} - \frac{80}{2} = 10,$$
$$d_3 = \frac{\alpha_3}{2\beta_3} - \frac{\gamma_{33}}{2\beta_3} = \frac{100}{2} - \frac{58}{2} = 21,$$

which gives a solution where production equals capacity in period 2 and 3, but where we have excess capacity of 10 units in period 1.

The solution above satisfies the capacity constraints (2.11) and the complementary slackness condition (2.12) of these constraints in the KKT-conditions, but, obviously, not the conditions (2.17)–(2.20) in Proposition 2.1 since the opportunity cost of producing the demands for period 2 and 3 in period 1 are respectively 22 and 24 (see Figure 1), which are much lower than 80 and 58 in (2.26) and (2.27). By this, it is evident that the excess capacity in period 1 of 10 units should be utilized as inventory from period 1 into for instance period 2. This is done by decreasing the disaggregated variable x_{22} by Δx and, at the same time, increasing (inventory) x_{12} with Δx . For period 2 at stage k, we make the transformation

$$x_{22}^{k+1} = x_{22}^{k} - \Delta x, \tag{2.28}$$
$$x_{12}^{k+1} = x_{12}^{k} + \Delta x. \tag{2.29}$$

In order to satisfy condition (2.17) in Proposition 2.1 at each step in the algorithm, this transformation can only be used when the opportunity costs of producing the demand d_2 in period 1 or 2 are equal, i.e., we first have to increase the multiplier for period 2 up to a level where $\gamma_{12} = \gamma_{22} = 22 \Rightarrow \Delta\lambda_2 = 2$. Increasing the multiplier of an infeasible period up to the level where the opportunity cost are equal is actually the main idea in the proposed *multiplier adjustment procedure* to solve our problem. But condition (2.17) also allows for more sophisticated transformations which are best described as *compositions* of the transformation (2.28) between several disaggregated variables. Such a composition is built up from the notion of a *link* between two disaggregated variables, which occurs exactly when the simple transformation (2.28) above is possible; i.e., when the opportunity costs equals.

Definition 2.2. (Link) A disaggregated variable x_{jit} is *linked* to another disaggregated variable $x_{ji't}$, $i \neq i'$ if the following conditions are satisfied:

$$\text{i) } x_{jit} > 0, \tag{2.30}$$
$$\text{ii) } \gamma_{jit} = \gamma_{ji't},$$
$$\text{iii) } \underline{r}_{i'} \geq 0. \tag{2.31}$$

A link is denoted as $x_{jit} \sim x_{ji't}$ and simply allows for using the transformation (2.28) of moving production of product j from period i to period i'. Condition i) assures that there is something to move from period i while ii) is the condition of equal opportunity costs between the periods as implied by condition (2.17) in Proposition 2.1. In condition iii), we require that there should not be infeasibility in the period i'. This condition is not required by Proposition 2.1 while being absolutely necessary since moving production from one period with infeasibility to another period which also has infeasibility does not make sense since we are trying to reduce the overall infeasibility.

As mentioned above, condition (2.17) allows for compositions of the transformation (2.28) and (2.29). Such composed transformations can only occur when one variable is linked to another which is then linked to another and so forth. The variables that are linked together are structured into a directed subgraph $G = (N, E)$ where the nodes N are the variables and the edges E occur 1) between linked variables and 2) between the variables adding up to the total production in a period.

2.4.2. Multiplier adjustment procedure. With the notion of links (and compositions) the proposed *multiplier adjustment procedure* can be described. At the first stage of the algorithm after initialization, all multipliers in subgraphs (periods) $G = (N, E)$ with infeasibility are increased *equally* by $\Delta\lambda$ up to a level where *either* i) capacity overload is *removed* in one (or more) subgraphs, *or* ii) up to a *minimum* level where the link or graph structure changes. If i) above occurs, the subgraph that becomes feasible is left out of the process of increasing the multipliers equally. If ii), we check whether there exists an excess capacity in the new subgraph. If so, this excess capacity is utilized in order to reduce or remove the capacity overload in the (one) overloaded period in the subgraph.

Note that, equally increasing the multipliers in *all* infeasible subgraphs implies that, if a new link is established, the added period or graph is automatically feasible, i.e., condition (2.31) is automatically satisfied by the algorithmic structure. By this, a graph will always contain at most *one* infeasible period.

Before presenting the algorithm the multiplier adjustment procedure is demonstrated by solving the single-item three-periodic example from above.

2.4.3. The three periodic example. The initialization step in our three-periodic (single-item) example gave capacity overloads in periods 2 and 3 by 30 and 19 capacity units, respectively (see Table 2). In step 1 (after initialization) the multipliers in both infeasible periods 2 and 3 are increased by $\Delta\lambda = 2$, i.e., up to a minimum level where the first link $x_{12} \sim x_{22}$ is established (see Figure 1). By (2.21) and (2.24), this increase in multipliers leads to the following demands and reduction in capacity overloads in the two periods:

$$
\begin{aligned}
d_2 &= \frac{\alpha_2}{2\beta_2} - \frac{\gamma_{22}}{2\beta_2} = \frac{100}{2} - \frac{22}{2} = 39,\\
d_3 &= \frac{\alpha_2}{2\beta_3} - \frac{\gamma_{22}}{2\beta_3} = \frac{100}{2} - \frac{22}{2} = 39,\\
\Delta\overline{r}_2 &= \frac{1}{2\beta_2}\Delta\lambda = \frac{1}{2}\cdot 2 = 1,\\
\Delta\overline{r}_3 &= \frac{1}{2\beta_3}\Delta\lambda = \frac{1}{2}\cdot 2 = 1.
\end{aligned}
$$

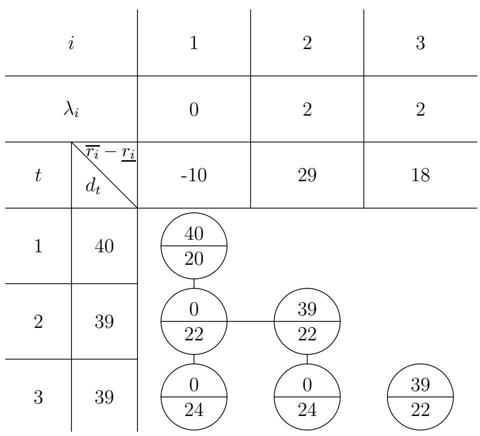

Figure 2. (Step 1): λ_1, λ_2 increased by 2.

Figure 2 illustrates the graph of our three-periodic example where periods 2 and 3 are linked together through $x_{12} \sim x_{22}$ (horizontal edge in the figure). When periods 1 and 2 are linked, the excess capacity in period 1 is utilized in order to reduce or remove the capacity overload in period 2. Figure 3 (step 2) shows this transformation where 10 capacity units in period 1 are utilized in order to store inventory of 10 product units from period 1 into period 2.

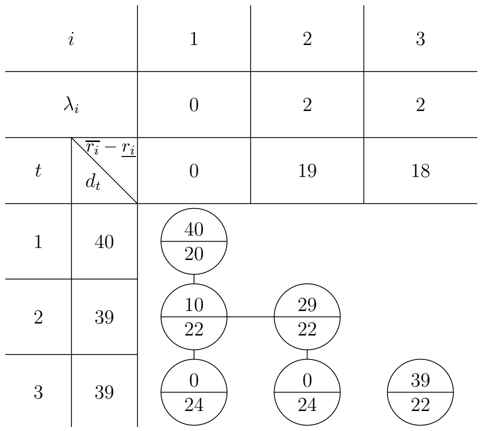

Figure 3. (Step 2): Utilize excess capacity in link $x_{12} \sim x_{22}$.

After step 2 is conducted, the periods 2 and 3 still have capacity overloads, while the capacity in period 1 is balanced. The only option to reduce or remove infeasibility in period 2 and 3 is then to increase the multipliers. The multipliers for the subgraph that cover link $x_{12} \sim x_{22}$ and the graph for period 3 are then increased equally until capacity overload is removed in one (or) both of the graphs. The necessary increase in $\Delta\lambda$ in order to remove the capacity overload of 19 units in period 2 is calculated by using (2.25).

$$\text{Period 2: } \Delta\lambda = \frac{\Delta\overline{r}_2}{\frac{1}{2\beta_1} + \frac{1}{2\beta_2}} = \frac{19}{\frac{1}{2} + \frac{1}{2}} = 19.$$

Similarly, the necessary increase in the multiplier in period 3 is calculated by

$$\text{Period 3: } \Delta\lambda = \frac{\Delta\overline{r}_3}{\frac{1}{2\beta_3}} = \frac{18}{\frac{1}{2}} = 36.$$

Since we reach feasibility in the subgraph covering period 1 and 2 first, we increase *all* multipliers with $\Delta\lambda = 19$ in step 3, which gives the solution shown in Figure 4.

It is important to note that the multiplier for period 1 is increased in step 3, although the capacity is balanced. The reason for this is that period 1 is linked with period 2. If only the multiplier for period 2 increases, then, by Proposition 2.1, the 10 units already stored from period 1 to 2 had to be shifted back to production in period 2 in order to satisfy the KKT-conditions. By increasing the multiplier equally we avoid this situation. The increase in the multiplier of 19 liberate capacity in period 1 by 9.5 units that can be utilized to produce 9.5 product units for period 2 which removes the capacity overload. It is worth noting that the solution in step 3 satisfies condition (2.17) in Proposition 2.1.

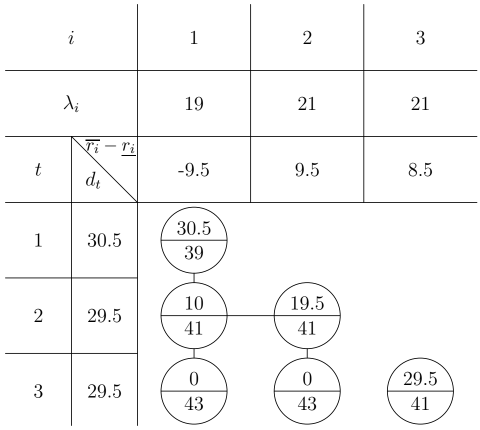

Figure 4. (Step 3): Increase $\lambda_{1,2,3}$ by 19.

The shifting of production from period 2 to period 1 by 9.5 units is shown in Figure 5 (step 4).

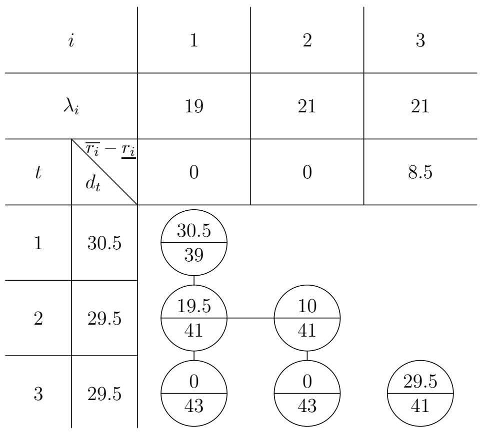

Figure 5. (Step 4): Utilize excess capacity in link $x_{12} \sim x_{22}$.

Since period 3 (after step 4) is the only period left with capacity overload (and the corresponding subgraph does not have excess capacity), the multiplier for period 3 is in step 5 increased up to a level where either period 3 is linked to period 1 or 2 or the overload is removed. In this case, period 3 links to period 2 at a value of $\Delta\lambda_3 = 2$, which occurs before the overload of 8.5 units is removed. The solution of this step is shown in Figure 6 (step 5) below.

After step 5, period 3 has still a capacity overload of 7.5 units. At the next stage (step 6) the periods 1, 2 and 3 form a composition where period 3 is linked with period 2 ($x_{33} \sim x_{23}$) and period 2 is linked with period 1 ($x_{22} \sim x_{12}$). Since no periods have excess capacity, we transform the the multipliers for all three periods by increasing tnem equally up to the level that removes the infeasibility in period

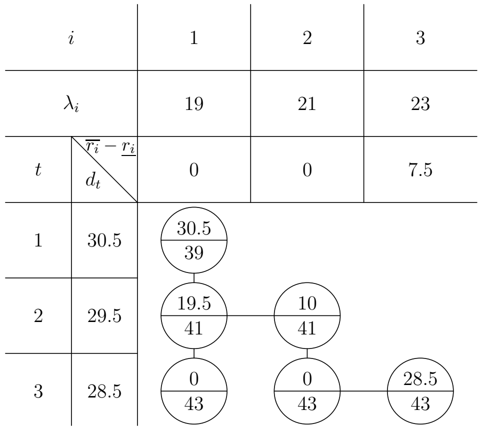

Figure 6. (Step 5): Increase of the multiplier in period 3, $\Delta\lambda_3 = 2$.

3. The necessary increase in the multiplier in order to remove this infeasibility is calculated as

$$\Delta\lambda = \frac{\Delta\overline{r}_3}{\frac{1}{2\beta_1} + \frac{1}{2\beta_2} + \frac{1}{2\beta_3}} = \frac{7.5}{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} = 5.$$

The increase in the multipliers of 5 gives, as shown in Figure 7, excess capacity in periods 1 and 2 of 2.5 capacity units in each.

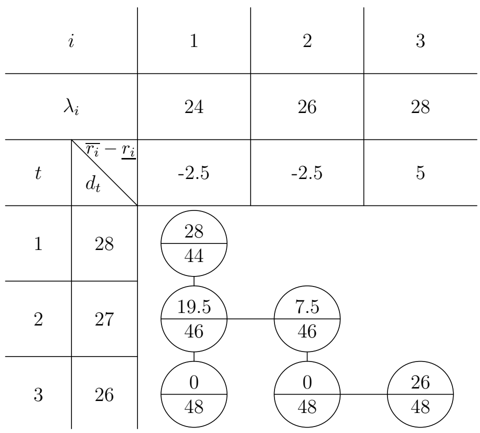

Figure 7. (Step 6): Multiplier increase of $\Delta\lambda = 5$ in periods 1, 2, 3.

The excess capacity of 2.5 units in both periods 1 and 2 is utilized to construct a feasible solution for the problem, which is shown in Figure 8. Note that many options exist for the value of the disaggregated variables that gives feasibility at this stage. In our solution (Figure 8), we have chosen $x_{12} = 22$ and $x_{23} = 5$. Choosing $x_{12} = 20$, $x_{23} = 3$ and $x_{13} = 2$ would obviously give the same solution.

It is easy to verify that the solution to our three-periodic problem satisfies the complete set of KKT-conditions.

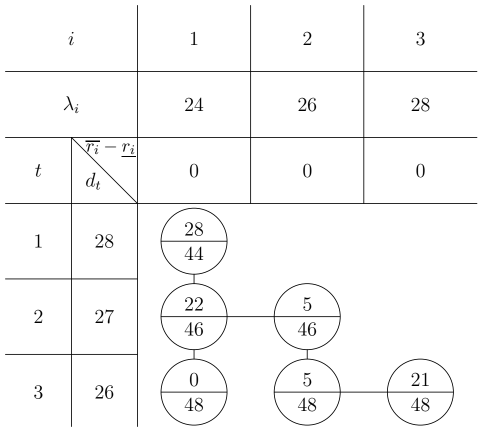

Figure 8. (Step 7): Utilize excess capacity in the link $x_{12} \sim x_{22} \sim x_{33}$.

The algorithm can now be formulated as

2.4.4. The algorithm.

(1) Start with $\lambda_0 = 0$.
(2) If $\overline{R}_i = \emptyset$, then stop.
If $\overline{R}_i \neq \emptyset$, then go to 3.
(3) Let k be the iteration number. Start with $k = 1$.
If there are subgraphs G with excess capacity and overload in period t, then set $\Delta\lambda^k = 0$ and utilize excess capacity by storing into period t until *either*
  i) capacity overload in period t is removed, *or*
  ii) excess capacity in the subgraph G is fully utilized, *or*
  iii) the graph-structure changes.
Else, increase λ^{k-1} in infeasible subgraphs *equally*, $\Delta\lambda^k > 0$, until *either*
  i) the first infeasibility is removed, *or*
  ii) up to the *minimum* level that changes the graph-structure.
Then go back to 2.

It is important to note that the graph-structure can change without increasing the multipliers (see (iii) at step 3). This can be explained by the following. Suppose in the single item case that an infeasible period t covers the demand for both periods t and t', $t' > t$, and the two periods are linked through the variables $x_{tt} \sim x_{tt'}$. Period t' is then by definition feasible. By the algorithm, $x_{tt'}$ will decrease. If $x_{tt'}$ becomes zero before the capacity overload in period t is removed, the link disappears or "dries out" (by condition (2.30)).

Throughout the process of increasing the multipliers, certain demands might be "priced out", i.e., in the single item case, this will happen if the opportunity costs $v_i + h_{it} + \lambda_i$ reach the level α_t, or in the multi-item case if $v_{ji} + h_{jit} + \lambda_i$ reach the level α_{jt}. If this happens, then these demands are left out from the problem.

The next section illustrates the proposed algorithm in multi-item case.

2.5. The multi-item case

The proposed algorithm applied to the multi-item case is demonstrated with a two-product, three-period problem. The data for this constructed problem are shown in Table 3 below.

t	1	2	3
p_{At}	$p_{A1} = 100 - 1d_{A1}$	$p_{A2} = 100 - 1d_{A2}$	$p_{A3} = 100 - 1d_{A3}$
x_{Ait}	x_{A11}	x_{A12}, x_{A22}	$x_{A13}, x_{A23}, x_{A33}$
$\gamma_{Ait} = v_{Ai}+h_{Ait}$	20	25, 20	30 25 20
p_{Bt}	$p_{B1} = 50 - 1d_{B1}$	$p_{B2} = 50 - 1d_{B2}$	$p_{B3} = 50 - 1d_{B3}$
x_{Ait}	x_{B11}	x_{B12}, x_{B22}	$x_{B13}, x_{B23}, x_{B33}$
$\gamma_{Bit} = v_{Bi}+h_{Bit}$	30	31 30	32 31 30
r_i	60	37	45

Table 3. A three-periodic problem with two products (problem data).

As can be seen in Table 3, the product demands are stationary (equal price functions for the three periods). Product A has a (constant) unit production and inventory holding cost of 20 and 5 respectively while these costs for product B are 30 and 1. The result from the initialization step where the demands for products A and B in all three periods are 40 and 10, respectively, are shown in Figure 9 below.

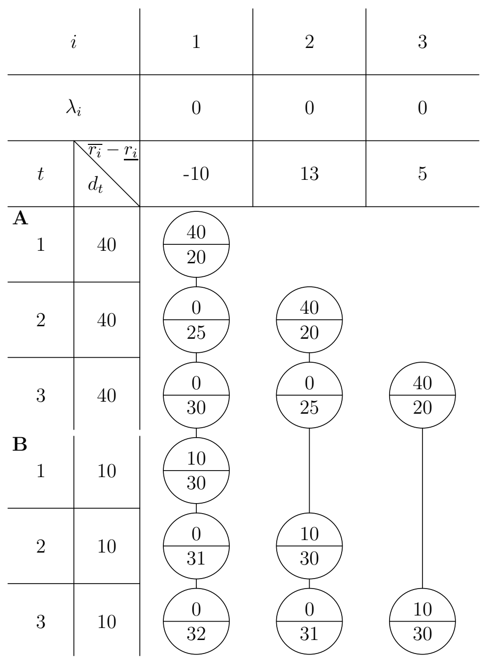

Figure 9. (Step 0): Initialization step — $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

The initialization step gives excess capacity of 10 units in period 1 while periods 2 and 3 have capacity overloads of 13 and 5 capacity units, respectively. Since there exists no excess capacity in the subgraphs that cover the infeasible periods 2 and 3, the multipliers for these two periods are increased equally up to the level where *either* the graph-structure changes *or* infeasibility is removed in one of the infeasible subgraphs covering period 2 and 3. As can bee seen in Figure 9, the disaggregated variable x_{B22} will be linked with x_{B12} when opportunity cost of producing the demand for product B in period 2 reach a level of 31, i.e., the level where the opportunity cost γ_{B22} equals the opportunity cost γ_{B12}. The minimum level of the allowed increase in the multipliers before the graph structure changes is, therefore, $\Delta\lambda = 1$. Before increasing the multiplier for periods 2 and 3 up to this level, we have to check whether infeasibilities in the two graphs are removed before we reach this level. The necessary increase in the multiplier for period 2 to remove the infeasibility of 13 capacity units is calculated by using (2.25)

$$\Delta\lambda_2 = \frac{\Delta\bar{r}_2}{\frac{1}{2\beta_{A2}} + \frac{1}{2\beta_{B2}}} = \frac{13}{1} = 13, \tag{2.32}$$

and, similarly, the necessary increase in the multiplier for period 3 is calculated by

$$\Delta\lambda_3 = \frac{\Delta\bar{r}_3}{\frac{1}{2\beta_{A3}} + \frac{1}{2\beta_{B3}}} = \frac{5}{1} = 5. \tag{2.33}$$

Since these necessary increases are above the minimum level $\Delta\lambda = 1$ where period 1 is linked with period 2 through product B, the multipliers for periods 2 and 3 are increased up to the level $\lambda_2 = \lambda_3 = 1$, ($\lambda_1 = 0$), which, by using (2.23), gives new demands for the two periods

$$\begin{aligned}
d_{A2} &= \frac{100}{2} - \frac{21}{2} = 39.5,\\
d_{A3} &= \frac{100}{2} - \frac{21}{2} = 39.5,\\
d_{B2} &= \frac{50}{2} - \frac{31}{2} = 9.5,\\
d_{B2} &= \frac{50}{2} - \frac{31}{2} = 9.5.
\end{aligned}$$

The results from step 1 where the multipliers in period 2 and 3 are increased by $\Delta\lambda = 1$ are shown in Figure 10.

After the increase in the multipliers in periods 2 and 3 at step 1, period 2 is linked with period 1 through product B and part of the excess capacity in period 1 is utilized by increasing the disaggregated variable $x_{B12} = 9.5$, i.e., up to the level where all the demand of product B is produced in period 1. This transformation is shown in Figure 11. Note that, when all the demands for B in period 2 is produced in period 1, the link $x_{B12} \sim x_{B22}$ will disappear (which is marked in Figure 11 by crossing out the link). When the link between periods 1 and 2 disappears, by Proposition 2.1, the multiplier for period 2 is allowed to be increased without increasing the multiplier for period 1. An increase in the multiplier for period 2 will only affect the demand (and price) for product A since all the demand for product B in period 2 is produced in period 1 ($x_{B22} = 0$).

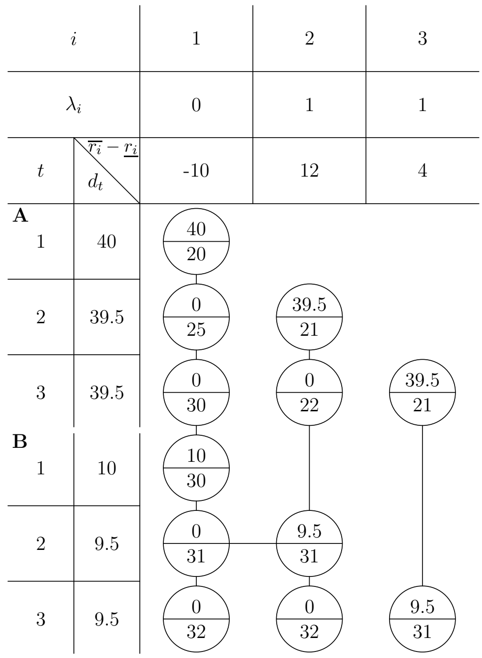

Figure 10. (Step 1): Increase multipliers in periods 2 and 3 ($\Delta\lambda = 1$).

After step 2, periods 2 and 3 still have capacity overloads of 2.5 and 4 capacity units, respectively while period 1 has an excess capacity of 0.5 units. Since no links exist between period 1 and the two other periods 2 and 3, the multipliers for period 2 and 3 are increased equally up to the level where either the graph-structure is changed or capacity overload in one or both periods are removed. From Figure 11 it can easily be verified that the graph-structure changes with $\Delta\lambda = 1$, i.e., when period 3 is linked with period 1 through product B. The necessary increase in the multipliers in order to remove infeasibility in the two graphs is calculated as $\Delta\lambda_2 = 5$ and $\Delta\lambda_3 = 4$, (by calculations similar to (2.32) and (2.33)). Thus, in step 3 (Figure 12), the multipliers for period 2 and 3 are increased with $\Delta\lambda = 1$, up to a level of 2 in both periods.

In step 4, the excess capacity of 0.5 units in period 1 is utilized by increasing x_{B13} to 0.5 units (and decreasing x_{B33} with the same quantity). This transformation is shown in Figure 13.

After step 4, period 2 and 3 still have capacity overload, while the capacity in period 1 is balanced. In order to remove infeasibility, the multipliers for all three periods are increased equally up to a level where the first graph (the graph covering period 1 and the graph covering periods 1 and 3) becomes feasible. The necessary increase in the multiplier for period 2 in order to remove the capacity overload of 2 capacity units is calculated by

$$\Delta\lambda_2 = \frac{\Delta\overline{r}_2}{\frac{1}{2\beta_{A2}}} = \frac{2}{\frac{1}{2}} = 4.$$

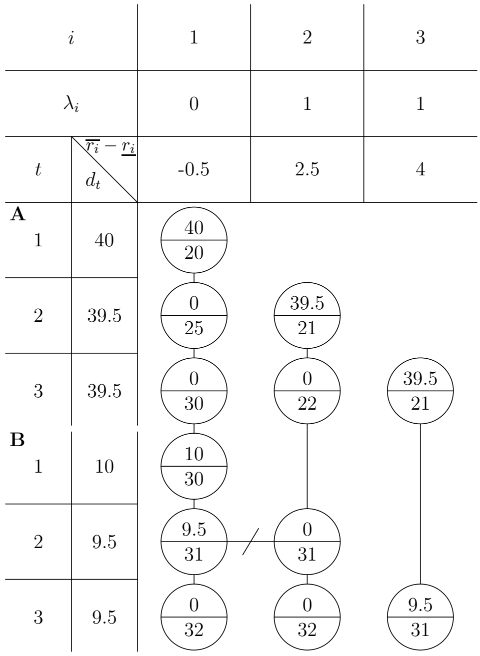

Figure 11. (Step 2): Utilize excess capacity in link $x_{B12} \sim x_{B22}$.

Similarly, the necessary increase in the multipliers in period 1 and 3 ($\Delta\lambda = \Delta\lambda_1 = \Delta\lambda_3$) in order to remove the infeasibility in period 3 of 2.5 capacity units is calculated by

$$\Delta\lambda = \frac{\Delta\overline{r}_3}{\frac{1}{2\beta_{A1}} + \frac{1}{2\beta_{B1}} + \frac{1}{2\beta_{B2}} + \frac{1}{2\beta_{A3}} + \frac{1}{2\beta_{B3}}} = \frac{2.5}{2.5} = 1.$$

By these calculations, the multipliers for all periods are increased by $\Delta\lambda = 1$ up to a level $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$. The result of these increases (step 5) is shown in Figure 14.

By increasing the multiplier for period 1 by $\Delta\lambda = 1$, the capacity in this period is liberated by 1.5 capacity units, which is utilized to increase the disaggregated variable x_{B13} with the same quantity. The result after this transformation (step 6) is shown in Figure 15.

After step 6, periods 1 and 3 are feasible, while period 2 has a capacity overload of 1.5 units. In order to remove the capacity overload in period 2, the multiplier for this period is allowed to increase up to the level where either period 2 is linked with the graph that covers period 1 and 2 or up to a (lower) level that removes the capacity overload. Figure 15 shows that period 2 will be linked with the graph for periods 1 and 3 when the multiplier λ_2 is increased by 3, i.e., up to a level where the opportunity cost of producing the demand for product A in period 1 equals the opportunity cost of producing this demand in period 1. The necessary increase in the multiplier in period 2 in order to remove the capacity overload of

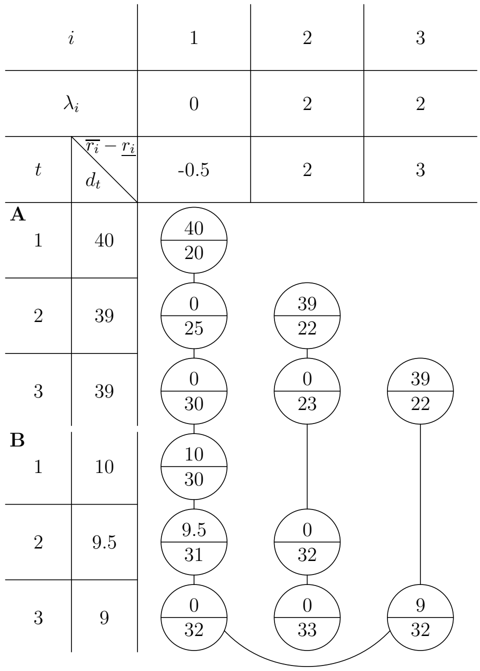

Figure 12. (Step 3): Increase in the multipliers for periods 2 and 3 with $\Delta\lambda = 1$.

1.5 units is calculated by

$$\Delta\lambda_2 = \frac{\Delta\overline{r}_2}{\frac{1}{2\beta_{A2}}} = \frac{1.5}{\frac{1}{2}} = 3,$$

i.e., exactly up to the level where period 2 is linked with period 1. The multiplier for period 2 is thereby increased by $\Delta\lambda_2 = 3$, which gives the solution shown in Figure 16.

It is easy to verify that the solution after step 7 satisfies the complete set of KKT-conditions (2.9)–(2.14).

2.6. Summary and further work

This paper describes an algorithm to solve a capacitated dynamic pricing problem. For the single item case, the proposed algorithm has $T\frac{(T+1)}{2}$ disaggregated variables that have to be updated at each stage of the algorithm. The proposed algorithm has a substantially greater complexity in the multi-item case since the number of disaggregated variables in the multi-item case is $JT\frac{(T+1)}{2}$. By this, the "curse of dimensionality" might be a problem if the number of planning periods is large. The algorithm applied to multi-item cases might, however, be justified by the fact that large scale problems can be solved effectively if the number of planning periods is low compared to the number of products. From a practical point of view, this justification seems to be reasonable – see 1.3.

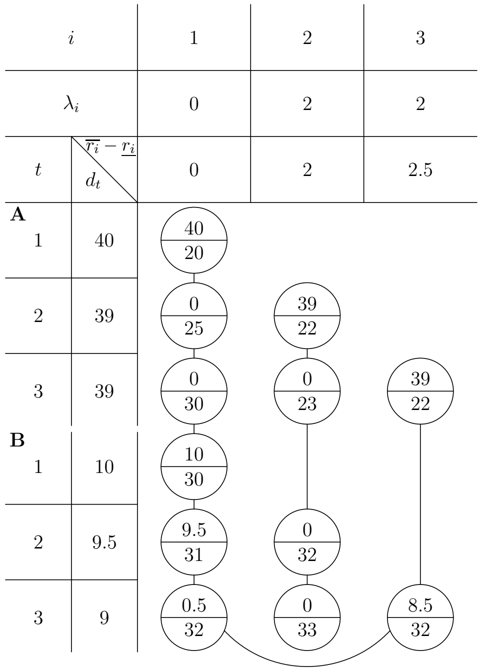

Figure 13. (Step 4): Utilizing excess capacity in the link $x_{B13} \sim x_{B33}$.

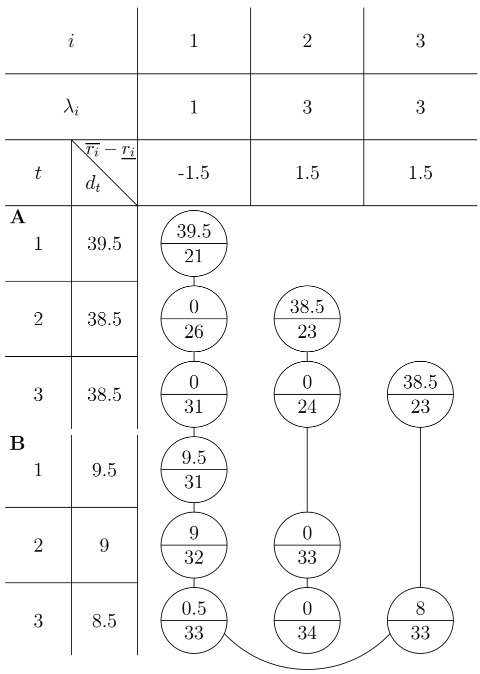

Figure 14. (Step 5): Increasing all multipliers by $\Delta\lambda = 1$.

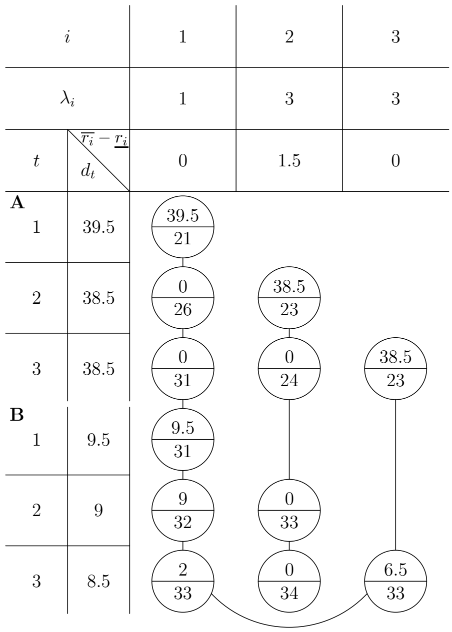

Figure 15. (Step 6): Utilize excess capacity in the graph.

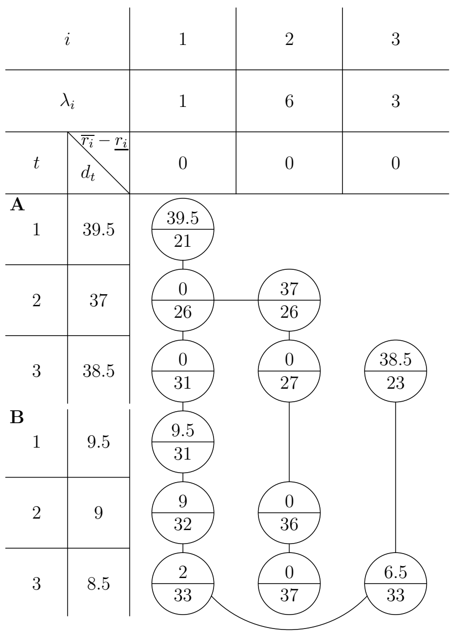

Figure 16. (Step 7): The multiplier for period 2 increased with $\Delta\lambda = 3$.

The proposed algorithms have not yet been implemented and tested. This is a topic for further work.

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